On the Relation of the Total Graph of a Ring and a Product of Graphs

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Abstract. The total graph of a ring $R$, denoted as $T(\Gamma(R))$, is defined to be a graph with vertex set $V\left(T\left(\Gamma(R)\right)\right) = R$ and two distinct vertices $u, v \in V\left(T\left(\Gamma(R)\right)\right)$ are adjacent if and only if $u + v \in Z(R)$, where $Z(R)$ is the zero divisor of $R$. The Cartesian product of two graphs $G$ and $H$ is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if: 1) $u_1 = u_2$ and $v_1 v_2 \in H$; or 2) $v_1 = v_2$ and $u_1 u_2 \in E(G)$. An isomorphism of graphs $G$ and $H$ is a bijection $\phi: V(G) \rightarrow V(H)$ such that $u, v \in V(G)$ are adjacent if and only if $\phi(u), \phi(v) \in V(H)$ are adjacent. This paper proved that $T\left(\Gamma\left(\mathbb{Z}_{2p}\right)\right)$ and $P_2 \times K_p$ are isomorphic for every odd prime $p$.

1. Introduction

Investigating group or ring properties and its structures from their graph representation become a new trend in graph theoretic research. Many authors proved that there are tight bonds between the rings and graphs. Aalipour in [1] investigated the chromatic number and clique number of a commutative ring. [2] gave a novel application of a central-vertex complete graph to a commutative ring. In 2008, [3] investigated the commutative graph of rings generated from matrices over a finite filed. Three years after the graph of a ring was introduced in [4], [5] proposed useful applications of semirings in mathematics and theoretical computer science. One interest in applying graph invariant on a group also showed in [6] in the properties of zero-divisor graphs. Another useful graph generated from group or ring structure is Cayley graphs which has many useful applications in solving and understanding a variety problem in several scientific interests [7].

The graph isomorphism itself has many applications in real life and many scientific fields [8]. [9] stated briefly about its application in the atomic structures and [10] showed how it can be applied in biochemical data. To prove the isomorphism of two graphs is an NP-problem in which there is no specific algorithm or certain way that works for all graphs in consideration [11]. In 1996, [12] proposed a good graph isomorphism algorithm but still troublesome for a large graphs.

Considering those applications of ring generated graphs, the applications of the graph isomorphisms, and the isomorphism-related algorithm complexity, finding an isomorphism of ring-structured graphs and the graph obtained from certain operation is a challenging task and a potential new interest in graph theory research. This paper considers the relation between the total graph of $\mathbb{Z}_p$ and $P_2 \times K_p$ for all odd prime $p$.

2. Preliminaries

A graph $G$ is a pair $G = (V, E)$ for non-empty set $V$ and $E \subseteq [V]^2$ (the elements of $E$ are 2-element subsets of $V$). For terminologies and notations concerning to a graph and its invariants, please consider [13]. This preliminary covers the definitions related to ring and the total graph of a ring. It also provides some definitions related to graph isomorphism and a graph operation.

**Definition 1. Ring** [14]

A ring $R$ is a set with two binary operations, addition and multiplication, such that for all $a, b, c \in R$:
1. $a + b = b + a$.
2. $(a + b) + c = a + (b + c)$.
3. There is an additive identity $0$.
4. There is an element $-a \in R$ such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$, and
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

With this definition, $\mathbb{Z}_{2p}$, an integer modulo $2p$ set, equipped with addition and multiplication modulo $2p$ operation is a ring.

**Definition 2. Zero Divisor** [14]

A zero-divisor is a nonzero element $a$ of a commutative ring $R$ such that there is a nonzero element $b \in R$ with $ab = 0$. 

100
Definition 3. Total Graph of a Ring [15]
Let $R$ be a ring and $Z(R)$ denotes the zero divisor of $R$. The total graph of $R$, denoted by $\Gamma(R)$ is an undirected graph with elements of $R$ as its vertices, and for distinct $x, y \in R$, the vertices $x$ and $y$ are adjacent if and only if $x + y \in Z(R)$.

From those definitions of zero divisor and total graph, we will construct a total graph of $\mathbb{Z}_{2p}$ for an odd prime $p$.

Definition 4. Graph Homomorphism and Isomorphism [13]
Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. A map $\varphi: V_G \to V_H$ is a homomorphism from $G$ to $H$ if it preserves the adjacency of the vertices. In another word, $(x, y) \in E_G \implies \{\varphi(x), \varphi(y)\} \in E_H$. If $\varphi$ is bijective and $\varphi^{-1}$ is also a homomorphism, then $\varphi$ is an isomorphism and $G$ is said to be isomorphic to $H$.

Definition 5. Cartesian Product [13]
The Cartesian product of two graphs $G$ and $H$ is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and two distinct vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if and only if: 1) $u_1 = u_2$ and $v_1v_2 \in E_H$; or 2) $v_1 = v_2$ and $u_1u_2 \in E(G)$. An isomorphism of graphs $G$ dan $H$ is a bijection $\phi: V(G) \to V(H)$ such that $u, v \in V(G)$ are adjacent if and only if $f(u), f(v) \in V(H)$ are adjacent.

3. Main Results

In this section we will prove the isomorphism of the total graph of $\mathbb{Z}_{2p}$ and $P_2 \times K_p$. We will investigate several properties of $\Gamma(\mathbb{Z}_{2p})$ before we proof the isomorphism. Those investigations will be provided as lemmas and theorems equipped with their proofs.

To characterize $\Gamma(\mathbb{Z}_{2p})$, we consider its vertex set, the degree of each vertex, and the clique it has as subgraphs, since $P_2 \times K_p$ can easily be considered and seen from those properties.

Lemma 1. The zero divisor of $\mathbb{Z}_{2p}$ is
$$Z(\mathbb{Z}_{2p}) = \{p\} \cup \{2n: n = 1, 2, ..., n - 1\}$$
for every odd prime $p$.

Proof.
For each $x \in \mathbb{Z}_{2p}$, the exactly one of the following holds: $x = p$, $x$ is even, and $x \neq p$ is odd.

Case 1. $x = p$
Since $2p = 0$ and $2 \in \mathbb{Z}_{2p}$, we conclude that $p \in Z(\mathbb{Z}_{2p})$.

Case 2. $x$ is even
Let $x = 2m$ for some $m \in \mathbb{Z}$. Since $xp = 2mp = m \cdot 2p = m \cdot 0 = 0$ and $p \in \mathbb{Z}_{2p}$, we conclude that $x \in Z(\mathbb{Z}_{2p})$ for all even $x \in \mathbb{Z}_{2p}$.

Case 3. $x \neq p$ is odd
If $x = 1$, then $xy \neq 0$ for all $0 \neq y \in \mathbb{Z}_{2p}$.
We will show that $1 \neq x \in Z(\mathbb{Z}_{2p})$ by using a contradiction. Suppose on the contrary, that $x \in Z(\mathbb{Z}_{2p})$. Consequently, there exists $0 \neq y \in \mathbb{Z}_{2p}$ such that $xy = 0$. It follows that $\gcd(x, 2p) > 1$. Since the factor of $2p$ is 2 and $p$, we obtain that $x$ divides $p$. It is a contradiction since $p$ is a prime number.
Lemma 2. Let \( p \) be an odd prime. Let \( A \subseteq \mathbb{Z}_{2p} \) be the set of all odd elements of \( \mathbb{Z}_{2p} \) and \( B \subseteq \mathbb{Z}_{2p} \) be the set of all even elements of \( \mathbb{Z}_{2p} \). \( \{u, v\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right) \) for all \( u, v \in A \) and \( \{x, y\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right) \) for all \( x, y \in B \). In another word, the vertices in \( A \) dan \( B \) form cliques in \( T\left(\Gamma(\mathbb{Z}_{2p})\right) \).

Proof.
Let \( u, v \in A \) and let \( u = 2s + 1 \) and \( v = 2t + 1 \) for some \( s, t \in \mathbb{Z} \). We obtain
\[
\begin{align*}
u + v &= 2s + 1 + 2t + 1 \\
&= 2(s + t + 1) \in Z(\mathbb{Z}_{2p}).
\end{align*}
\]
Therefore \( \{u, v\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right) \) for all \( u, v \in A \).

Let \( x, y \in A \) and let \( x = 2s \) and \( y = 2t \) for some \( s, t \in \mathbb{Z} \). We obtain
\[
\begin{align*}
x + y &= 2s + 2t \\
&= 2(s + t) \in Z(\mathbb{Z}_{2p}).
\end{align*}
\]
Therefore \( \{x, y\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right) \) for all \( x, y \in B \).

It proves that \( A \) and \( B \) form cliques in \( T\left(\Gamma(\mathbb{Z}_{2p})\right) \).

Lemma 3. Let \( A \) and \( B \) be sets defined in Lemma 2 and \( p \) be an odd prime number. For each \( v \in A \) there is a unique \( x \in B \) such that
\[
\{v, x\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right).
\]

Proof.
For each \( v \in A \), choose \( x = p - v \). It can be easily verified that \( x \in B \) since \( p \) and \( v \) are both odd numbers. On the other hand, let \( x \in B \) and \( x \neq p - v \). Suppose that \( \{v, x\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right) \), that is \( v + x \in Z(\mathbb{Z}_{2p}) \). Since \( v \) is odd and \( x \) is even, it follows that \( v + x \) is an odd number and \( v + x = p \iff x = p - v \), a contradiction. This proves that \( \{v, p - v\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right), \forall x \in A \).

Analogous to this proof, we can easily prove that for each \( x \in B \) there is a unique \( v \in A \) such that
\[
\{v, x\} \in E\left(T\left(\Gamma(\mathbb{Z}_{2p})\right)\right).
\]

Before we discuss the main problem, consider Figure 1 that represents the graph \( T\left(\Gamma(\mathbb{Z}_{2p})\right) \) for several \( p \).

Figure 1. \( T\left(\Gamma(\mathbb{Z}_{2p})\right) \) for \( p \in \{3,5,7\} \).
Theorem 1. For any odd prime \( p \), \( T \left( \Gamma \left( \mathbb{Z}_{2p} \right) \right) \) is isomorph to \( P_2 \times K_p \).

Proof. Let \( V(P_2) \) and \( V(K_p) \) be labeled as \( \{p_1, p_2\} \) and \( \{k_0, k_1, \ldots, k_{p-1}\} \) respectively. The vertices of the resulting graph obtained from the Cartesian product, \( P_2 \times K_p \), is therefore labeled
\[
\{(p_1, k_1), (p_1, k_2), \ldots, (p_1, k_p), (p_2, k_1), (p_2, k_2), \ldots, (p_2, k_p)\}
\]
in which
\[
\{(p_s, k_i), (p_s, k_j)\} \in E(P_2 \times K_p), \forall i, j \in \{1, 2, \ldots, p\}
\]
and \( i \neq j \), for \( s \in \{1, 2\} \). Other edges to consider is \( \{(p_1, k_i), (p_2, k_{(p-i \mod p)+1})\} \in E(P_2 \times K_p), \forall i \in \{1, 2, \ldots, p\} \). Here, the “mod” in “\( p - i \ mod \ p \)” is a modulus operator, not a modulus relation.

Consider the function \( \varphi: V \left( T \left( \Gamma \left( \mathbb{Z}_{2p} \right) \right) \right) \to V(P_2 \times K_p) \) defined as follows:
\[
\varphi(x) = \begin{cases} 
(p_1, \left(\frac{p - x}{2}\right) \mod p + 1), & \text{if } x \text{ is odd} \\
(p_2, \frac{x}{2} + 1), & \text{if } x \text{ is even.}
\end{cases}
\]

Since \( \varphi \) is a bijective function that preserves adjacency of the vertices of \( V \left( T \left( \Gamma \left( \mathbb{Z}_{2p} \right) \right) \right) \) and \( V(P_2 \times K_p) \), we conclude that \( T \left( \Gamma \left( \mathbb{Z}_{2p} \right) \right) \) and \( P_2 \times K_p \) are isomorphic.

Figure 1 and Figure 2 show some examples of the mapping result of \( \varphi \).

4. Conclusion

From the discussion, we conclude that \( T \left( \Gamma \left( \mathbb{Z}_{2p} \right) \right) \) and \( P_2 \times K_p \) are isomorphic.
References


